Uitwerking Tentamen Kwantumfysica 1
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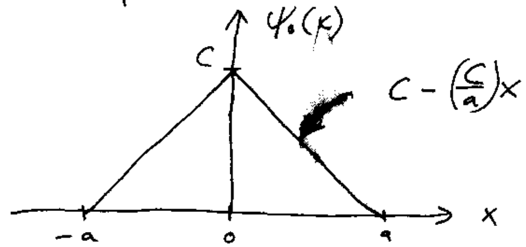
Problem 1

a) Approximate wavefunction for example with triangular shape

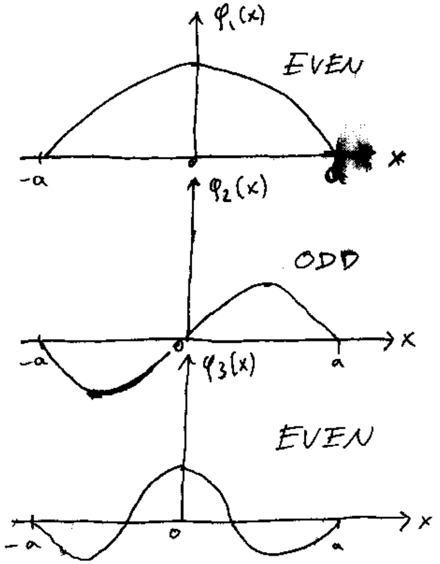


Wave function is normalized $\Rightarrow \int_{-a}^{a} |\psi_0(x)|^2 dx = 1 \Rightarrow$

$2\int_0^a \left|C-\left(\frac{C}{a}\right)x\right|^2 dx = 1 \Rightarrow 2C^2\left[x-\frac{x^2}{a}+\frac{x^3}{3a^2}\right]_0^a = 1 \Rightarrow$

$C=\sqrt{\frac{3}{2a}}$ $\qquad a = 1\cdot 10^{-9}\ m \Rightarrow C=\sqrt{1{,}5\cdot 10^{9}}\cdot m^{-1/2}$

b)



See book eq. (6.100)

$\sqrt{\frac{2}{2a}}\cos\left(\frac{\pi x}{2a}\right)$

$\sqrt{\frac{2}{2a}}\sin\left(\frac{2\pi x}{2a}\right)$

$\sqrt{\frac{2}{2a}}\cos\left(\frac{3\pi x}{2a}\right)$

c) $\psi_0(x) = \psi_0(-x) \Rightarrow$ Parity is EVEN

(Function is symmetric about $x=0$). Book p. 177

<u>Note:</u> this means that $\psi_0(x)$ must be a superposition of <u>only</u> <u>even</u> energy eigenfunctions $\varphi_n(x)$ ($\Rightarrow$ only with <u>odd</u> n)

d) Work towards expression in <u>x-representation</u>! (read the question well!)

$W(E_n) = |\langle\psi_0|\varphi_n\rangle|^2 = \left|\int_{-a}^{a} \varphi_n^*(x)\psi_0(x)\,dx\right|^2$

for $n = 1, 2, 3$

<u>Note:</u> Is $|$overlap integral of $\varphi_n(x)$ with $\psi_0(x)|^2$

Result is 0 for $\varphi_n(x)$ an ODD function.

Result >0 for $\varphi_n(x)$ an EVEN function

e) $W(E_2)$ is zero (see d))

$\psi_0(x)$ resembles $\varphi_1(x)$ more than $\varphi_3(x)$ $\Rightarrow$

$W(E_1) > W(E_3) > W(E_2) = 0$

f) $W(E_2)=0$, see d), e)

$W(E_1)$: Use again triangle approximation as for a). for $\psi_0(x)$

$\Rightarrow W(E_1) = \left|2\int_0^a \left(C-\left(\frac{C}{a}\right)x\right)\cdot\sqrt{\frac{2}{2a}}\cos\left(\frac{\pi x}{2a}\right)dx\right|^2$

(See postulate **5**, hoorcollege week 2)

This integral can be solved exactly, by hand, but is a bit of work $\Rightarrow$

$$W(E_1) = \frac{64\,a\,C^2}{\pi^4} = \frac{9^6}{\pi^4} = 0.985$$

Alternatively, you can solve the integral graphically: Draw both $\psi_0(x)$ and $\varphi_1(x)$, then sketch the product. Determine the area, and take this squared $\Rightarrow$ also gives an estimate that $W(E_1)$ is in the range 0.5 to 1. $W(E_1)$ is quite high since $\psi_0(x)$ resembles $\varphi_1(x)$.

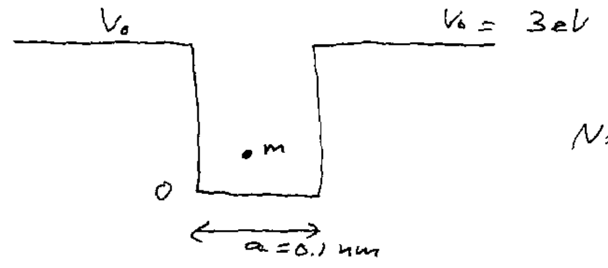
$$W(E_3) = \left| 2\int_0^a \left(C - \left(\frac{C}{a}\right)x\right)\sqrt{\frac{2}{2a}}\cos\left(\frac{3\pi x}{2a}\right)dx \right|^2 \Rightarrow$$

Solve exact gives $W(E_3) = \frac{64\,a\,C^2}{81\pi^4} = \frac{32}{27\pi^4} = 0.012$

Graphical solution with $W(E_3)$ in range 0 to 0.5 is also good answer. $\varphi_3(x)$ resembles $\psi_0(x)$ less than $\varphi_1(x)$.

Problem 2

a)



$N = 1$

See book p. 278 and further, on finite potential well. Solution here uses the same notation, but note that the book has a well with width $2a$, here it is a.

$n=3$ Third energy eigen state is a bound state

$n=4$ Fourth energy eigen state is not a bound state

$\boxed{\text{Use here } b = \frac{a}{2}}$

eigenstates $\Rightarrow \xi^2 + \eta^2 = \rho^2 = \frac{2mb^2|V|}{\hbar^2}$

Third eigenstate exists for $\pi < \rho < \frac{3}{2}\pi$

Fourth eigen state exists for $\frac{3}{2}\pi < \rho < 2\pi$ $\Rightarrow$

System has three bound states for

$$\pi < \sqrt{\frac{2mb^2|V_0|}{\hbar^2}} < \frac{3}{2}\pi \Rightarrow$$

$$\frac{\hbar^2\pi^2}{2b^2|V_0|} < m < \frac{9\hbar^2\pi^2}{8b^2|V_0|} \Rightarrow$$

For

$0{,}137\cdot 10^{-27}\,kg < m < 0{,}154\cdot 10^{-27}\,kg$

the system has 3 bound states.

b) & c)

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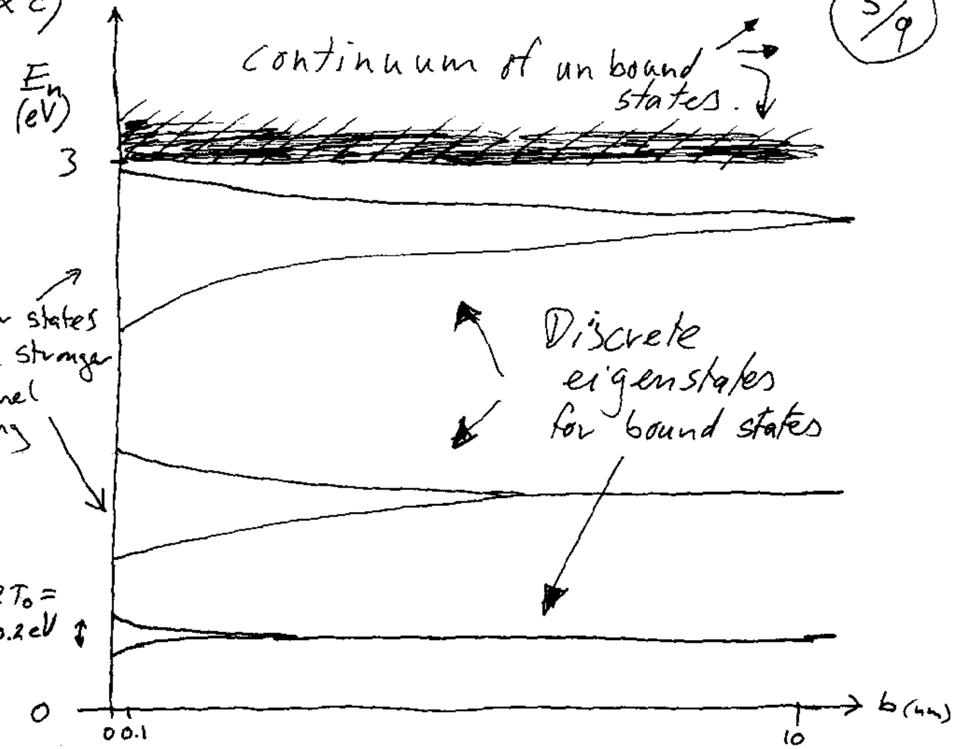
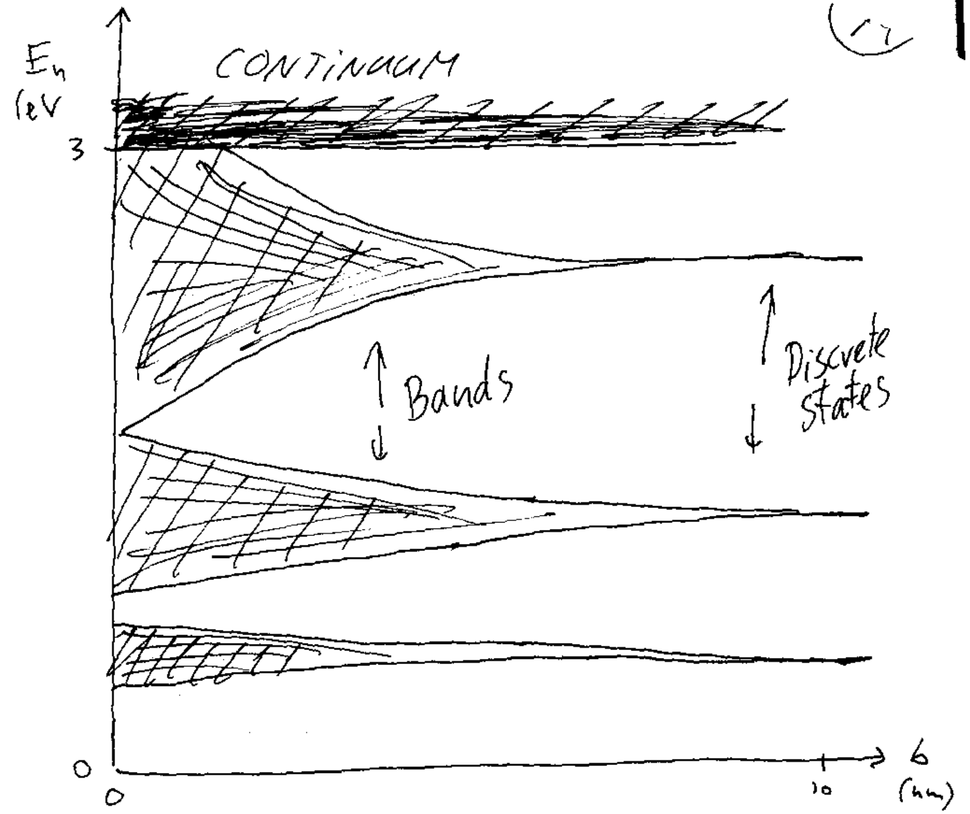


d) Continuum of states, so level spacing is <u>zero</u>.

e)



See book p. 303 & p. 316 and hoorcollege week 7

f) For example electrons in periodic potential formed by solid state latice.

Problem 3

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a) $\hat{H} = \hbar\omega_0\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$ with $\hat{N} = \hat{a}^\dagger\hat{a}$

b) $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{K}{2}\hat{x}^2$

with $\omega_0^2 = \frac{K}{m}$

c) Show that $\langle\Psi_n|\Psi_n\rangle = 1$

$$\langle\Psi_1|\Psi_1\rangle = \left|i\sqrt{\tfrac{1}{2}}\right|^2 + \left|\sqrt{\tfrac{3}{8}}\right|^2 + \left|-i\sqrt{\tfrac{1}{8}}\right|^2$$

$$= \frac{1}{2} + \frac{3}{8} + \frac{1}{8} = 1$$

$$\langle\Psi_2|\Psi_2\rangle = \left|\sqrt{\tfrac{1}{3}}\right|^2 + \left|i\sqrt{\tfrac{1}{3}}\right|^2 + \left|-i\sqrt{\tfrac{1}{3}}\right|^2$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

d) All terms $\langle\varphi_m|\varphi_n\rangle$ with $m \neq n$ are zero
with $m = n$ are 1

$$\langle\Psi_1|\Psi_2\rangle = \left(i\sqrt{\tfrac{1}{2}}\right)^*\sqrt{\tfrac{1}{3}}\langle\varphi_0|\varphi_0\rangle + 0$$

$$= -i\sqrt{\tfrac{1}{6}}$$

Note: calculating $\langle\Psi_2|\Psi_1\rangle = \langle\Psi_1|\Psi_2\rangle^*$ is also ok.

e) Time evolution of system in state $|\Psi_2\rangle$ is

$$|\Psi(t)\rangle = \hat{U}|\Psi_2\rangle = e^{\frac{-i\hat{H}t}{\hbar}}|\Psi_2\rangle$$

Emission of radiation is related to oscillations of the dipole in time. Whether the dipoles oscillates, and at what frequencies, can be determined by evaluating

$$\langle D(t)\rangle = \langle\Psi(t)|\hat{D}|\Psi(t)\rangle$$

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In this case

$$|\Psi(t)\rangle = \hat{U}|\Psi_2\rangle = \sqrt{\tfrac{1}{3}}e^{-\frac{i}{2}\omega_0}|\varphi_0\rangle + i\sqrt{\tfrac{1}{3}}e^{-\frac{5i}{2}\omega_0}|\varphi_2\rangle - i\sqrt{\tfrac{1}{3}}e^{-\frac{9i}{2}\omega_0}|\varphi_4\rangle$$

The expression for $\langle D(t)\rangle$ will have terms in the form (a summation over these terms)

$$e^{+\frac{2i}{2}(n\omega_0 - m\omega_0)}\langle\varphi_n|\hat{D}|\varphi_m\rangle$$

- which are stationary (constant, no oscillations) for $n = m$
- and terms that oscillate for $n \neq m$ at a angular frequency of $(n-m)\,\omega_0$

Note that $\langle\varphi_n|\hat{D}|\varphi_m\rangle$ is not zero for $n \neq m$, because $\hat{D} \propto \hat{x}$, and $\hat{x}$ does not commute with $\hat{H}$.

So the system in state $|\Psi_2\rangle$ at $t = 0$ will indeed oscillate and radiate, with angular frequencies $\underline{2\omega_0}$ and $\underline{4\omega_0}$.

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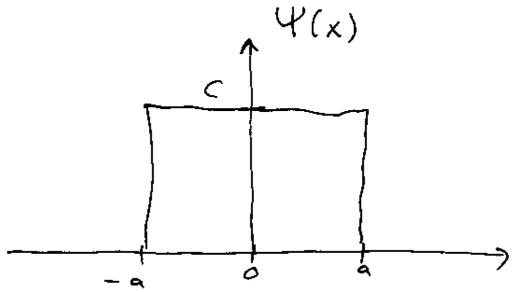
a) $\Psi(x)$



$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} c\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-c}{ik} e^{-ikx} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{-c}{ik} e^{-ika} - \frac{c}{ia} e^{+ika} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{c}{k} \left(2 \sin(ka) \right) \Rightarrow$$

$$\boxed{\overline{\Psi}(k) = \frac{2ca}{\sqrt{2\pi}} \frac{\sin(ka)}{ka}}$$

b) It is a free particle, so $\hat{H}$ is not a function of $\hat{x}$ $\Rightarrow$ $[\hat{H}, \hat{x}] = 0 \Rightarrow \frac{d\langle k \rangle}{dt} = 0$.

$\overline{\Psi}(k)$ does not change in time.

(No forces on the particle that can change momentum $p = \hbar k$)